Foucault pendulum through basic geometry

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(Received 4 April 2006; accepted 14 June 2007)

We provide a thorough explanation of the Foucault pendulum that utilizes its underlying geometry on a level suitable for science students not necessarily familiar with calculus. We also explain how the geometrically understood Foucault pendulum can serve as a prototype for more advanced phenomena in physics known as Berry's phase or geometric phases.

[DOI: 10.1119/1.2757623]

I. INTRODUCTION

Since Léon Foucault's public demonstration of his pendulum experiment, it has played a prominent role in physics, physics education, and the history of science. The Foucault pendulum is a long pendulum suspended high above the ground and carefully set into planar motion. The phenomenon described by Foucault[1] concerns the orientation of the plane of oscillation of the pendulum. We define the *orientation of the pendulum* as the orientation of its plane of oscillation.[2] An observer on Earth witnesses that the orientation of the pendulum (a) slowly rotates during the course of the day, and (b) in general does not return to its original orientation after one day. Although related, these two observations address conceptually different aspects of the phenomenon. We refer to (a) as the dynamic aspect and to (b) as the geometric aspect.

The difference in the initial and final orientation of the pendulum is called the *phase shift* and is the focus of this article (cf. Fig. 1). We define a positive phase shift as a shift in the counterclockwise direction. Because the phase shift is an angle, it is defined up to addition or subtraction of 2π (a "full turn").

We will derive a relation for the phase shift and show how this result applies to other physical phenomena. The only approximations made are that any friction forces are isotropic, that is, they do not prefer any direction, the motion of the pendulum at any time is planar, and the plane of oscillation of the pendulum passes through the center of the Earth as gravity exerts its pull on the pendulum. Note that we explicitly allow most friction forces that would be relevant in experimental settings, and we will see that they have no effect on the phase shift. The assumption of planar motion is sometimes called the assumption of adiabaticity and is justified because the period of the pendulum is small compared to the period of the rotation of Earth. This assumption underpins all calculations of the Foucault pendulum.

The last condition is not satisfied if the centrifugal force is taken into consideration. From the perspective of an inertial observer the centrifugal force enters because the suspension point of the pendulum follows a circular path. However, the centrifugal force is small compared to gravity and has little effect on the phase shift. To account for the centrifugal force we can replace the third condition by assuming that the plane of oscillation of the pendulum contains the plumb line. We will discuss this more accurate physical model in the Appendix.

We explicitly allow for large oscillations, as long as the pendulum oscillates but does not rotate.[3] The explanation presented here is not new. In fact, we merely elaborate on the following paragraph from Wilczek and Shapere:[4]

> "How does the pendulum precess when it is taken around a general path C? For transport along the equator, the pendulum will not precess. [...] Now if C is made up of geodesic segments, the precession will all come from the angles where the segments of the geodesics meet; the total precession is equal to the net deficit angle which in turn equals the solid angle enclosed by C modulo 2π. Finally, we can approximate any loop by a sequence of geodesic segments, so the most general result (on or off the surface of the sphere) is that the net precession is equal to the enclosed solid angle."

There are some published explanations of the Foucault pendulum along similar arguments (see for example, Hart, Miller, and Mills[5] and Opera[6]). These explanations require at least vector calculus and differential equations. Also, they approach the Foucault pendulum from the dynamic perspective and do not arrive at the insight that the phase shift depends only on global properties of the path.

In this paper we reduce the arguments to basic geometry, making the argument more accessible and at the same time more general so that we can understand the essence of what is driving the phase shift. The prerequisite for our approach is an understanding of some geometry and trigonometry, as well as knowing the following relation for the area $S(B_h)$ of a "belt" of height h on a sphere of radius r (see Fig. 2):

$$S(B_h) = 2\pi r h. \tag{1}$$

See Ref. 8 for a classical proof of Eq. (1). Modern treatments of the proof are shorter, but require basic calculus.

II. MATHEMATICAL AND PHYSICAL MODEL

To understand the Foucault pendulum geometrically, we consider an inertial frame outside of Earth, in which the center of Earth is fixed and Earth is rotating about the axis through the poles.[7] In this way we may view Earth as rotating on top of an imaginary sphere, which is fixed in space. We record the motion of the pendulum with respect to this fixed sphere.

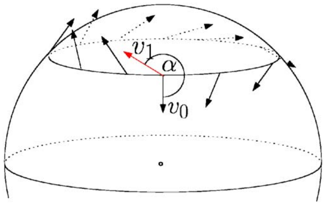

Fig. 1. The orientation of the plane of oscillation slowly rotates during the course of the day, in general not returning to its original orientation $\mathbf{v_0}$, but resulting in a final orientation $\mathbf{v_1}$ differing from $\mathbf{v_0}$ by an angle α.

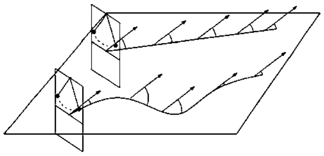

Fig. 3. The pendulum as a compass. When an oscillating pendulum is picked up at its suspension point and moved in the plane, the angle its plane of oscillation makes with a straight line path remains constant, but the angle it makes with a curved path changes.

From this perspective it appears that the suspension point of the pendulum traces out a closed path on the sphere in one day, as Earth is rotating on top of it. At different times during the day, the pendulum is situated at different points on this sphere, which makes it difficult to compare the orientation of the pendulum at these times. After one day the pendulum returns to its original position on the sphere, and we can compare the initial and final orientations to obtain the phase shift.

In accordance with our assumptions we ignore the centrifugal force (see the Appendix for the treatment including the centrifugal force). From the perspective of our inertial frame, there are no external forces that could cause the orientation of the pendulum to change—gravity acts radially. The orientation of the pendulum is constrained to remain tangent to the sphere because the plane of oscillation of the pendulum passes through the center of the sphere. We see that the orientation of the pendulum is governed by free motion (no external forces), subject to the constraint that the orientation remain tangent to the sphere.

III. THE PHASE SHIFT

In this section we derive the relation for the phase shift of the orientation of the Foucault pendulum after one day. To gain a deeper understanding of the phenomenon it is useful to consider a pendulum whose suspension point traverses arbitrary paths on the sphere, not just paths of fixed latitudes. The result for the classical Foucault pendulum will be a special case, but the more general result provides deeper geometric insight. We will use the results obtained here to understand other seemingly unrelated physical systems, an example of which is given in Sec. IV.

Before we discuss the Foucault pendulum, we take another look at the pendulum in the Euclidean plane. A pendulum that is slowly taken along a path in the Euclidean plane can serve as a compass. A symmetry argument shows that the orientation of the pendulum remains fixed as the pendulum is taken along a path: gravity acts downward, so it cannot turn the orientation of the pendulum which is constrained to lie in the plane. In the language of physics this constraint corresponds to the conservation of the direction of the angular momentum. If the path is a straight line, the angle between the path and the orientation of the pendulum remains constant. If the path is not straight, the orientation of the pendulum still does not change, but the direction of the path does, so the angle the orientation makes with the path changes (see Fig. 3).

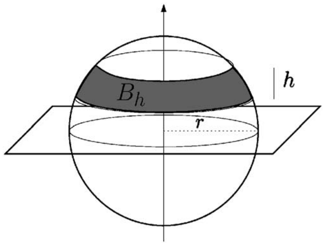

Fig. 2. A "belt" of height h on the sphere. Archimedes had a diagram with a sphere and cylinder of same height and diameter inscribed on his tombstone. He was the first to show that the two figures have the same area by proving that the area of the belt of height h on the sphere has the same area as a cylinder of the same radius as the sphere and height h.

This simple observation turns out to be crucial for understanding the motion of a pendulum on the sphere, that is, the Foucault pendulum. The pendulum on a sphere behaves in exactly the same way.

On the sphere the point of view of the pendulum as a compass is complicated by the fact that there is no absolute notion of orientation tangent to the sphere (see for example Theorem 2.28 in Ref. 14). Without the notion of absolute orientation we cannot *a priori* compare orientations at different points on the sphere. But, we can still compare orientations along paths, as illustrated in Fig. 3. We start by investigating the notion of straight lines on the sphere.

A. Straight lines on the sphere

To understand which paths on the sphere should be called "straight" we resort to Newton's first law and define a straight line to be the path a particle travels in the absence of external forces. In our case the particles we consider are constrained to stay on the surface of the sphere.

The equator is an example of a straight line on the sphere. It divides the sphere into two mirror symmetric halves. In the absence of external forces, a particle set in motion along the equator cannot distinguish the two hemispheres and thus has to stay on the equator. Of course the equator is not a straight line in three dimensions. If we were to allow the particle to leave the surface of the sphere, it would travel along a different trajectory.

By the same argument, the image of the equator after a rigid rotation of the sphere is also a straight line. These paths on the sphere are called *great circles* or geodesics. Alternatively, we may think of great circles as the intersection locus of the sphere with planes that pass through the center of the sphere.

We have established that every great circle is a straight line on the sphere. Conversely, we show that every straight line on the sphere is a great circle. Recall that the path of a particle is uniquely determined by its initial position and initial direction of travel. For any such position and direction, there is a great circle passing through the position and

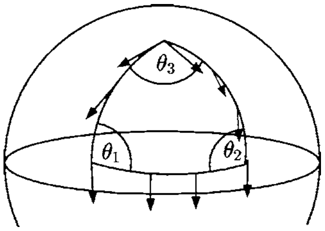

Fig. 4. A triangle on the sphere. All sides are segments of great circles. When a pendulum is taken along a triangular path, the angle the orientation of the pendulum makes with the great circle segments remains constant along each segment. Thus, only the angles contribute to the phase shift.

tangent to the given direction and thus the particle must follow that great circle. In particular, paths of fixed latitude other than the equator are not straight lines.

When the pendulum is taken around a great circle, the angle the orientation makes with the great circle never changes. This phenomenon can again be understood by a symmetry argument, because the great circle divides the sphere into two indistinguishable parts. Therefore, the change in the angle cannot favor either of the two halves, and thus the angle cannot change at all.

If however the pendulum is taken along a path that is not a great circle, for example a path of fixed latitude other than the equator, then the angle between the orientation of the pendulum and the path does change.

We conclude that the direction given by a great circle yields an inertial frame for orientations along that path, whereas the direction given by other paths does not. In particular, the directions given by latitudes and longitudes do not form an inertial frame along paths of fixed latitude other than the equator.

B. Triangles on the sphere

Now consider a triangular path on the sphere, that is, a path consisting of three great circle segments as in Fig. 4.

Let θ_1, θ_2, and θ_3 be the angles at the vertices of the triangle. We start somewhere along the great circle segment between the angles θ_3 and θ_1. Assume for simplicity that the initial orientation of the pendulum is parallel to this segment. When traversing this segment, the angle the orientation makes with the great circle segment does not change—the sides of the triangle are segments of great circles. At the vertex of the angle θ_1, we start using the second great circle segment as a reference. The angle of the orientation with respect to the first segment was zero, but with respect to the second segment the angle is $\theta_1 - \pi$. This angle does not change as we traverse the second great circle segment. Similarly, the orientation will acquire an additional angle of $\theta_2 - \pi$ at the second vertex (as we start using the third segment as reference), and another angle of $\theta_3 - \pi$ at the third vertex, when we again use the original segment as reference. In total the orientation has acquired a phase shift $\alpha(C)$ given by

$$\alpha(C) = (\theta_1 - \pi) + (\theta_2 - \pi) + (\theta_3 - \pi) \tag{2}$$

$$= \theta_1 + \theta_2 + \theta_3 - \pi, \tag{3}$$

remembering that the phase shift is only defined modulo 2π.

In the Euclidean plane, the sum of the angles of a triangle equals π so the phase shift is zero; the proof is similar to the previous argument. For a spherical triangle the sum of the angles is always greater than π because the sphere has positive curvature.

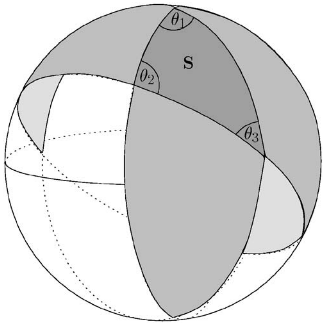

Fig. 5. The figure obtained after cutting the sphere along the great circles making up the sides of the triangle and gluing together the original triangle and the three neighboring pieces.

There is a fundamental geometric result relating the phase shift for a triangular path C on a sphere of radius r to the area $S(C)$ of the triangle,

$$\alpha(C) = \frac{S(C)}{r^2} \quad \text{(Gauss-Bonnet theorem for the sphere)}. \tag{4}$$

The proof we present is due to Euler; see also the discussion in McCleary.[9] Fix a triangular path C on the sphere and cut the sphere along the great circles defined by the sides of the triangle, separating the sphere into 8 pieces. Take the piece consisting of the original triangle, and reattach the three neighboring pieces to obtain a new figure (see Fig. 5). The leftover pieces can be assembled in a similar way, forming the mirror image of that figure.

The figure covers exactly half of the sphere and its mirror image covers the other half. So, the area of the figure is $2\pi r^2$, half the area of the sphere which is $4\pi r^2$ by Eq. (1) with $h = 2r$.

Now look at the three lunes, that is, the "2-angles" in Fig. 5, given by the original triangle together with a choice of only one of the three neighboring pieces. The two vertices lie on antipodal points of the sphere, and the angle at the vertices is just the corresponding angle θ in the original triangle. The lune with angle θ covers a portion of $\theta/2\pi$ of the sphere. So, the lune has area $2\theta r^2$, using again that the sphere has area $4\pi r^2$.

If $S(C)$ is the area of the original triangle, then the surface area of the figure, which is $2\pi r^2$, can also be calculated as the sum of the areas of the three lines minus $2S(C)$, because we have counted the area of the triangle three times. Thus,

$$2\theta_1 r^2 + 2\theta_2 r^2 + 2\theta_3 r^2 - 2S(C) = 2\pi r^2, \tag{5}$$

or equivalently

$$S(C) = (\theta_1 + \theta_2 + \theta_3 - \pi) r^2. \tag{6}$$

So, for any triangular path C on the sphere we can combine Eqs. (3) and (6) to obtain Eq. (4), completing the proof.

The quantity $S(C)/r^2$ is also called the *enclosed solid angle*. We assume that the path C enclosing the area $S(C)$ is traversed in the positive (counterclockwise) direction; otherwise we count the area $S(C)$ as negative.

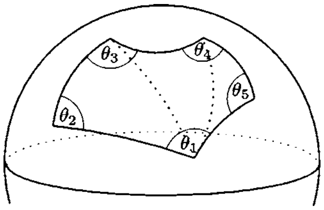

Fig. 6. Polygon on the sphere. All segments are great circles. The polygon can be subdivided into several triangles.

C. General paths

Now we generalize the result to a path C which is a polygon (see Fig. 6). Any such polygon can be divided into triangles, and the phase shift acquired when traversing C equals the sum of the phase shifts associated with each triangle. Similarly, the area of the polygon equals the sum of the areas of the triangles. Therefore Eq. (4) remains valid in this case.

Now let C be an arbitrary closed path. We may approximate C by polygons. The phase shift depends continuously on the path and so does the enclosed area. Therefore Eq. (4) also holds in the general case, proving the Gauss-Bonnet theorem for all closed paths C on the sphere. We see that the Gauss-Bonnet theorem provides the desired geometric interpretation of the phase shift. The difference between the initial and final orientations of the pendulum when taken around a closed path depends only on the area the path encloses on the sphere.

D. The Foucault pendulum

We now return to the special case of paths of fixed latitude λ. This case is the one relevant for calculating the phase shift for a Foucault pendulum situated at a latitude λ. We use Eq. (1) with $h=r-r\sin(\lambda)$ and find that the area enclosed by this path is

$$S(C_\lambda)=2\pi r^2(1-\sin(\lambda)), \tag{7}$$

so the phase shift is given by

$$\alpha(C_\lambda)=\frac{S(C_\lambda)}{r^2}=2\pi[1-\sin(\lambda)]=-2\pi\sin(\lambda), \tag{8}$$

remembering that the phase shift is defined modulo 2π. Equation (8) is the classical result for the Foucault pendulum (see, for example, Arnol'd[10]).

So far we have only addressed the geometric aspect of the Foucault pendulum. For completeness we use our results to understand how the orientation of the pendulum changes in time. To this end we need to be able to compare angles at different points along the path of fixed latitude λ. We keep track of the angle $\theta(t)$ the pendulum makes with the path at time t, measured in days. From the perspective of an earthbound observer this is natural, because the direction of the path corresponds to a frame fixed on Earth as Earth rotates.

By symmetry, the angle θ changes equal amounts in equal time, so if $\theta(0)=\theta_0$ is the orientation at time $t=0$, then $\theta(t)-\theta_0=ct$ for some constant c depending on λ. We know the phase shift after one day, so $c=\theta(1)-\theta_0=\alpha(C_\lambda)$; the equality holds modulo 2π. If we use Eq. (8), we see that $c=-2\pi\sin(\lambda)+2\pi k$, where k is an integer.

In Sec. III A we saw that if the pendulum is on the equator ($\lambda=0$), the angle the orientation makes with the equator does not change, so $c=0$ on the equator. Therefore $k=0$, and

$$\theta(t)=-2\pi\sin(\lambda)t+\theta_0, \tag{9}$$

which is the dynamic expression for the Foucault pendulum at latitude λ when viewed in an earthbound reference frame.

IV. THE FOUCAULT PENDULUM AS A PROTOTYPE OF A GEOMETRIC PHASE

The principle underlying the phase shift of the Foucault pendulum appears in other physical phenomena, ranging from electrodynamics to nuclear physics, quantum mechanics, deformable bodies, and fractional statistics. We refer the interested reader to Refs. 4 or 11. In the following we briefly describe a simple application from optics.[12]

Consider a laser emitting linearly polarized light that is fed into an optical fiber that winds in space in some fashion; the light exits the fiber in the same direction as it enters. Assume that the momentum vector of the light is always parallel to the fiber, and that there are no forces due to torsional stress of the fiber. We want to understand the phase shift in the direction of polarization when comparing the initial direction of polarization to the direction after the light exits the fiber, in relation to the path of the fiber.

In the semiclassical approximation we view the direction of polarization as a vector perpendicular to the momentum vector. Again, there are no external forces acting on the polarization, but it satisfies a constraint. The speed of light is constant, so the momentum vector moves on a sphere in momentum space as the fiber traces out its path. Because the direction of the exiting light was assumed to be the same as the one of the entering light, the momentum vector traces out a closed path on a sphere, and the polarization vector is constrained to remain tangent to it.

Described in this way, we see that this system acts similarly as the Foucault pendulum, and therefore the phase shift follows the same rule; it is given by the Gauss-Bonnet relation applied to the path in momentum space. For example, if the fiber follows a two-dimensional path, the initial and final directions of polarization coincide. In this case the momentum vector is confined to a great circle, and so the enclosed solid angle is a multiple of 2π. If the fiber follows a three-dimensional path, the direction of polarization may pick up a nontrivial phase shift.

V. CONCLUSION

Describing the Foucault pendulum geometrically allowed our considerations to be very general, for example, allowing for friction and large oscillations of the pendulum. By generalizing the argument to arbitrary paths we found that the Foucault relation has a deep geometric meaning and is independent of the local properties of the path—it depends only on the enclosed area. This independence is an insight that is lost in dynamic derivations of the motion of the pendulum. Moreover, we explained how the Foucault pendulum can be seen as a prototype of a wide range of phenomena in various branches of physics.

Many authors, (see for example, the list in Ref. 5) including Foucault himself (as quoted by Aczel[15]), have asserted that "the plane of motion of the pendulum remains fixed as Earth rotates underneath the pendulum." This statement is false, because the pendulum does not return to its original orientation after one day (except at the poles and the equator). But there is some correct intuition that can be extracted from this statement when the effect is understood geometrically.

The insight we have gained in this paper suggests the following correction. The phrase "remains fixed" should not be interpreted as remains fixed in 3-space, but instead as "remains fixed in an inertial frame tangent to Earth," and the phrase "as earth rotates underneath the pendulum" should be replaced by "as the path of fixed latitude bends underneath the pendulum," or "as the coordinate frame given by latitudes and longitudes rotates underneath the pendulum."

When explaining the Foucault pendulum geometrically to students who are not familiar with spherical geometry, we have found it useful to use hands-on tools to demonstrate the arguments given in Sec. III. The tools are exemplified using manipulatives such as a large ball and a "south-pointing chariot" (a mechanical compass). When taken around a path of fixed latitude on a ball, the orientation given by the south-pointing chariot behaves exactly like the orientation of the Foucault pendulum. For more details on this use see Ref. 13.

ACKNOWLEDGMENTS

We thank Carl Pennypecker and the reviewers for corrections and helpful suggestions.

APPENDIX: THE EFFECT OF THE CENTRIFUGAL FORCE

If we assume that the plane of oscillation of the pendulum contains the plumb line rather than the center of Earth, the orientation of the pendulum is constrained to be perpendicular to the plumb line instead of being perpendicular to the radial direction. This altered constraint is related to the fact that Earth is not really a sphere but an oblate spheroid, whose surface is perpendicular to the plumb line. Thus, just like for the sphere, the orientation of the pendulum is now constrained to be tangent to the oblate spheroid.

We solve the question of the pendulum on the oblate spheroid by relating it to the pendulum on the sphere. Given any point on the oblate spheroid, we associate to it a point on the sphere by choosing the unique point on the sphere that has the same local normal. This process is called a *Gauss map*.[9] When a pendulum traces out a path on the oblate spheroid, the Gauss map associates this path with a path on the sphere. Because the orientation of the pendulum is constrained to be perpendicular to the local normal at all times, and the local normals on the oblate spheroid and the associated points on the sphere are the same (by definition), the two pendula behave in exactly the same way—they feel the same constraint.

To calculate the Foucault relation for a path of fixed latitude on the oblate spheroid, we need to understand what latitude means in this setting. There are several reasonable ways to define latitude on the oblate spheroid. The choice that is most common—and also most convenient for our situation—is the *geographic latitude*, that is, the angle the local vertical makes with the equatorial plane.

The Gauss map takes a path of fixed geographic latitude λ on the oblate spheroid and associates with it a path of fixed latitude λ on the sphere. Thus, geographic latitude is really the Gauss map in disguise, and the Foucault relation is still valid in the setting of the oblate spheroid.

In summary, we have two different models, a pendulum on a sphere at latitude λ, and a pendulum on the oblate spheroid at geographic latitude λ, both with the constraint that the orientation of the plane of oscillation be tangent to the surface. In both cases, the same relation holds.

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[1] M. L. Foucault, "Démonstration physique du mouvement de rotation de la terre au moyen du pendule," C. R. Acad. Sci. Hebd Seances Acad. Sci. D **32**, 135 (1851).

[2] During a period of the pendulum, the angular momentum changes its magnitude continuously and flips to its opposite direction twice. The plane perpendicular to it is well defined, except at the singular times when the angular momentum is zero. In an experimental setting it is more natural to use the direction of the swing of the pendulum as the orientation rather than the perpendicular direction as we do. For the theoretical treatment in this article our choice is more convenient.

[3] If the pendulum has high energy and rotates around its suspension point without coming to a rest and turning around, we would have to take gyroscopic forces into consideration. In the case where the pendulum oscillates back and forth these forces average out.

[4] *Geometric Phases in Physics*, edited by F. Wilczek and A. Shapere (World Scientific, Singapore, 1989).

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[7] As Foucault noted (Ref. 1) a nonrotating frame comoving with the Earth is not an inertial frame either, because the Earth is rotating around the Sun, and the Sun is rotating around the center of the Milky Way. On the time scale of one day, we may neglect these effects and treat the motion of the center of the Earth as straight line motion.

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